

Renormalized thermodynamic entropy of black holes in higher dimensions

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We study the ultraviolet divergent structures of the matter (scalar) field in a higher D -dimensional Reissner-Nordström black hole and compute the matter field contribution to the Bekenstein-Hawking entropy by using the Pauli-Villars regularization method. We find that the matter field contribution to the black hole entropy does not, in general, yield the correct renormalization of the gravitational coupling constants. In particular, we show that the matter field contribution in odd dimensions does not give the term proportional to the area of the black hole event horizon. [S0556-2821(97)06104-3]

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I. INTRODUCTION

One of the salient problems concerning Bekenstein-Hawking [1,2] black hole entropy is to understand its microscopic origin. As an attempt to understand this problem 't Hooft suggested that the thermodynamic entropy of a scalar field coupled to a black hole background can give rise to the correct black hole entropy, provided that a suitable brick wall is introduced outside the event horizon [3]. The entropy of the matter field diverges as the brick wall approaches to the event horizon, and Susskind and Uglum [4] suggested that the matter field contributions be interpreted as one-loop corrections to classical Bekenstein-Hawking entropy and renormalize the gravitational coupling constants. Demers, Lafrance, and Meyers [5] (DLM) have subsequently confirmed, by using the Pauli-Villars covariant regularization method, the Susskind-Uglum conjecture for the scalar field minimally coupled to a four-dimensional Reissner-Nordström (RN) black hole background.

DLM have shown that in four dimensions the renormalization constants arising in the renormalization of black hole entropy are precisely the same as those arising in the renormalization of the gravitational action by a quantum (zero temperature) scalar field. They used the Pauli-Villars regularization by assuming that all the regulator fields obey the Bose-Einstein statistics.

To further clarify the question of the matter field contributions we study, in this paper, the ultraviolet-divergent structures of a massive scalar field in a higher

D -dimensional RN black hole, compute the matter field contribution to the black hole entropy using the Pauli-Villars regularization method, and compare the results with the divergent terms arising in the one-loop renormalization of the gravitational action.

The thermodynamic entropy of a massive scalar field in the D -dimensional RN black holes was already found in the brick wall model [6], in which the RN black hole is linearized as the Rindler form by considering a pillbox-shaped region close to the outer event horizon and the number of states of the scalar field in the Rindler metric is used to compute the entropy. The ζ -function regularization scheme was also used to find the divergent structure and renormalized entropy of the scalar field in a D -dimensional Rindler-like spacetime [7]. In this paper, however, we consider the scalar field in the D -dimensional RN black hole without approximating it by the Rindler form near the outer event horizon, find alternatively the number of states from the semiclassical quantization of the Klein-Gordon equation, and get the divergent structure of the free energy and entropy as the divergent functions of brick wall thickness.

In Sec. II we elaborate, by using the Pauli-Villars regularization method, the one-loop renormalization of the D -dimensional gravitational action by a quantum scalar field. In the next three sections we study the ultraviolet-divergent structures of the scalar field and compute the scalar field contribution to the D -dimensional RN black hole entropy by using the Pauli-Villars regularization method with all the regulator fields treated as obeying Bose-Einstein statistics. We find in particular that the entropy contribution of the scalar field minimally coupled to an odd dimensional RN black hole background does not give the term proportional to the surface area of the event horizon. We conclude with some discussions in the last section. Two appendixes contain

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some mathematical formulas needed for the computations in the main text.

Throughout this paper we adopt the units $c = \hbar = k = 1$, but keep the gravitational constant G . The spacetime signature is $(-, +, \dots, +)$.

II. RENORMALIZATION OF GRAVITATIONAL ACTION BY QUANTUM FIELDS

Quantum field theory has been extensively studied in curved spacetimes (for references, see [8]). Dimensional regularization, ζ -function regularization, point-splitting, and Pauli-Villars regularization methods have been developed to find the renormalized effective action for a quantum field. We shall use the Pauli-Villars method that enables us to regularize the thermodynamic entropy of the quantum field in a black hole background and to relate this directly to the renormalization of the action. We shall further elaborate the Pauli-Villars regularization method, which is efficient in evaluating the effective action and the thermodynamic entropy.

The one-loop effective action for gravity in D dimensions [8] takes the form

$$\mathcal{I}_D = \int d^D x \sqrt{-g} \left[-\frac{\Lambda}{8\pi G} + \frac{R}{16\pi G} + \frac{\alpha_1}{4\pi} R^2 + \frac{\alpha_2}{4\pi} R_{\mu\nu} R^{\mu\nu} + \frac{\alpha_3}{4\pi} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right], \quad (1)$$

where Λ is the cosmological constant, G the gravitational constant, and α_i the coupling constants. These constants are bare ones that are to be renormalized. The spacetime that we are particularly interested in is a black hole background minimally coupled to a massive scalar field. The one-loop effective action of the scalar field of a mass m can be found by the DeWitt-Schwinger method [8,9]:

$$W_D(m) = \frac{1}{2(4\pi)^{D/2}} \int d^D x \sqrt{-g} \int_0^\infty d(is) \sum_{k=0}^\infty a_k(x) \times (is)^{k-1-D/2} e^{-im^2 s}, \quad (2)$$

where

$$a_0 = 1, \quad a_1 = \frac{1}{6} R,$$

$$a_2 = \frac{1}{30} R_{;\mu}^{;\mu} + \frac{1}{72} R^2 + \frac{1}{180} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu}. \quad (3)$$

The effective action W_D involves divergent terms from the lower limit $s=0$. According to the Pauli-Villars regularization method, we introduce a number of bosonic and fermionic regulator fields of masses m_{B_i} and m_{F_i} , respectively. All the regulator and scalar fields contribute to the effective action

$$W_D = \frac{1}{2(4\pi)^{D/2}} \int d^D x \sqrt{-g} \int_0^\infty d(is) \sum_{k=0}^\infty a_k(x) \times (is)^{k-1-D/2} \left(\sum_i e^{-im_{B_i}^2 s} - \sum_i e^{-im_{F_i}^2 s} \right). \quad (4)$$

The effective action obtained by the dimensional regularization method shows different divergent structures depending on even or odd dimensions. In the Pauli-Villars regularization method the divergent structure of the even dimensional case differs from that of the odd dimensional case.

A. Even dimensions

We first consider even dimensions. Using the analytical continuation of the integral (A10) in Appendix A, we obtain the divergent contributions to the effective action in an even dimension $D = 2n$:

$$\begin{aligned} W_{2n}^{\text{div}} = & \frac{1}{2(4\pi)^n} \int d^{2n} x \sqrt{-g} \sum_{k=0}^n a_k(x) (-1)^{n+1-k} \left[\frac{1}{(n-k)!} \left(\sum_i m_{B_i}^{2(n-k)} \ln(m_{B_i}^2) - \sum_i m_{F_i}^{2(n-k)} \ln(m_{F_i}^2) \right) \right. \\ & - \frac{1}{(n-k)!} \left(I_1 + \sum_{p=1}^{n-k} \frac{1}{p} \right) \left(\sum_i m_{B_i}^{2(n-k)} - \sum_i m_{F_i}^{2(n-k)} \right) \\ & \left. + \sum_{l=2}^{n+1-k} \frac{(-1)^l}{(n+1-k-l)!} I_l \left(\sum_i m_{B_i}^{2(n+1-k-l)} - \sum_i m_{F_i}^{2(n+1-k-l)} \right) \right]. \end{aligned} \quad (5)$$

To remove the infinite constants I_p given by Eq. (A11), we impose the mass conditions

$$\sum_i m_{B_i}^{2(n-k)} = \sum_i m_{F_i}^{2(n-k)} \quad (6)$$

for $k=0, 1, \dots, n$. We are then left with the renormalized action

$$W_{2n}^{\text{ren}} = \int d^{2n} x \sqrt{-g} \sum_{k=0}^n a_k(x) \frac{\mathcal{B}_k}{2(4\pi)^n (n-k)!}, \quad (7)$$

where

$$\mathcal{B}_k = (-1)^{n+1-k} \left(\sum_i m_{B_i}^{2(n-k)} \ln(m_{B_i}^2) - \sum_i m_{F_i}^{2(n-k)} \ln(m_{F_i}^2) \right) \quad (8)$$

are the renormalization constants. We may now renormalize the one-loop effective action for gravity by quantum matter field by redefining the cosmological, gravitational, and coupling constants

$$\begin{aligned} \frac{\Lambda}{8\pi G} - \frac{\mathcal{B}_0}{2(4\pi)^n n!} &= \frac{\Lambda^{\text{ren}}}{8\pi G^{\text{ren}}}, \\ \frac{1}{16\pi G} + \frac{\mathcal{B}_1}{12(4\pi)^n (n-1)!} &= \frac{1}{16\pi G^{\text{ren}}}, \\ \frac{\alpha_1}{4\pi} + \frac{\mathcal{B}_2}{144(4\pi)^n (n-2)!} &= \frac{\alpha_1^{\text{ren}}}{4\pi}, \\ \frac{\alpha_2}{4\pi} - \frac{\mathcal{B}_2}{360(4\pi)^n (n-2)!} &= \frac{\alpha_2^{\text{ren}}}{4\pi}, \\ \frac{\alpha_3}{4\pi} + \frac{\mathcal{B}_2}{360(4\pi)^n (n-2)!} &= \frac{\alpha_3^{\text{ren}}}{4\pi}. \end{aligned} \quad (9)$$

For instance, in $D=4$ the conditions are such that the number of bosonic and fermionic fields are equal,

$$N_B = N_F = 3, \quad (10)$$

and the masses satisfy

$$\begin{aligned} \sum_{i=1}^3 m_{B_i}^2 &= \sum_{i=1}^3 m_{F_i}^2, \\ \sum_{i=1}^3 m_{B_i}^4 &= \sum_{i=1}^3 m_{F_i}^4. \end{aligned} \quad (11)$$

One may choose a simple solution [5]

$$\begin{aligned} m_{B_1} &= m, \quad m_{B_2} = m_{B_3} = \sqrt{m^2 + 3\mu^2}, \\ m_{F_1} &= m_{F_2} = \sqrt{m^2 + \mu^2}, \quad m_{F_3} = \sqrt{m^2 + 4\mu^2}. \end{aligned} \quad (12)$$

The renormalization constant related to the gravitational constant can be written as

$$\begin{aligned} \mathcal{B}_1 &= m^2 \ln \left(\frac{m^2(m^2 + 3\mu^2)^2}{(m^2 + \mu^2)^2(m^2 + 4\mu^2)} \right) \\ &+ \mu^2 \ln \left(\frac{(m^2 + 3\mu^2)^6}{(m^2 + \mu^2)^2(m^2 + 4\mu^2)^4} \right) \end{aligned} \quad (13)$$

and becomes, for a large μ ,

$$\mathcal{B}_1 \simeq m^2 \ln \left(\frac{3^2 m^2}{4\mu^2} \right) + 2\mu^2 \ln \left(\frac{3^3}{2^4} \right). \quad (14)$$

The other renormalization constant is

$$\mathcal{B}_2 = -\ln \left(\frac{m^2(m^2 + 3\mu^2)^2}{(m^2 + \mu^2)^2(m^2 + 4\mu^2)} \right), \quad (15)$$

which becomes, for a large μ ,

$$\mathcal{B}_2 \simeq -\ln \left(\frac{3^2 m^2}{4\mu^2} \right). \quad (16)$$

In $D=6$, one has an additional condition on the masses,

$$\sum_i m_{B_i}^6 = \sum_i m_{F_i}^6, \quad (17)$$

one of whose simple solutions is

$$\begin{aligned} m_{B_1} &= m, \quad m_{B_2} = \sqrt{m^2 + 3\mu^2}, \quad m_{B_3} = \sqrt{m^2 + 4\mu^2}, \\ m_{B_4} &= \sqrt{m^2 + 7\mu^2}, \quad m_{F_1} = m_{F_2} = \sqrt{m^2 + \mu^2}, \\ m_{F_3} &= m_{F_4} = \sqrt{m^2 + 6\mu^2}. \end{aligned} \quad (18)$$

We rewrite $\mathcal{B}_1$ as

$$\begin{aligned} \mathcal{B}_1 &= -m^4 \ln \left(\frac{m^2(m^2 + 3\mu^2)(m^2 + 4\mu^2)(m^2 + 7\mu^2)}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^2} \right) \\ &- 2m^2 \mu^2 \ln \left(\frac{(m^2 + 3\mu^2)^3(m^2 + 4\mu^2)^4(m^2 + 7\mu^2)^7}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^{12}} \right) \\ &- \mu^4 \ln \left(\frac{(m^2 + 3\mu^2)^9(m^2 + 4\mu^2)^{16}(m^2 + 7\mu^2)^{49}}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^{72}} \right). \end{aligned} \quad (19)$$

For a large μ it becomes

$$\mathcal{B}_1 \simeq -m^4 \ln \left(\frac{7m^2}{3\mu^2} \right) - 2m^2 \mu^2 \ln \left(\frac{7^7}{2^4 3^9} \right) - \mu^4 \ln \left(\frac{7^{49}}{2^{40} 3^{63}} \right). \quad (20)$$

The other two renormalization constants are

$$\begin{aligned} \mathcal{B}_2 &= m^2 \ln \left(\frac{m^2(m^2 + 3\mu^2)(m^2 + 4\mu^2)(m^2 + 7\mu^2)}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^2} \right) \\ &+ \mu^2 \ln \left(\frac{(m^2 + 3\mu^2)^3(m^2 + 4\mu^2)^4(m^2 + 7\mu^2)^7}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^{12}} \right) \end{aligned} \quad (21)$$

and

$$\mathcal{B}_3 = -\ln \left(\frac{m^2(m^2 + 3\mu^2)(m^2 + 4\mu^2)(m^2 + 7\mu^2)}{(m^2 + \mu^2)^2(m^2 + 6\mu^2)^2} \right). \quad (22)$$

It should be noted that $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ can be regarded as independent for an arbitrary large μ , since they involve different powers of μ . That is, there is a unique representation of $\ln(\mu^2)$, μ^2 , and μ^4 in terms of $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$.

B. Odd dimensions

In odd dimensions we use similarly the integral formula (A23) in Appendix A. We then obtain the divergent contributions to the effective action in an odd dimension $D=2n+1$:

$$W_{2n+1}^{\text{div}} = \frac{1}{2(4\pi)^{(2n+1)/2}} \int d^{2n+1}x \sqrt{-g} \sum_{k=0}^n a_k(x) (-1)^{n+1-k} \left[\frac{2^{2(n+1-k)} \pi^{1/2} (n+1-k)!}{(2n+2-2k)!} \left(\sum_i m_{B_i}^{2n+1-2k} \ln(m_{B_i}^2) \right) \right. \\ \left. - \sum_i m_{F_i}^{2n+1-2k} \ln(m_{F_i}^2) \right) + \sum_{l=1}^{n+1-k} \frac{(-1)^l}{(n+1-k-l)!} I_{(2l+1)/2} \left(\sum_i m_{B_i}^{2(n+1-k-l)} - \sum_i m_{F_i}^{2(n+1-k-l)} \right) \right]. \quad (23)$$

Imposing again the same mass conditions (6) as in $D=2n$, we get the renormalized effective action

$$W_{2n+1}^{\text{ren}} = \int d^{2n}x \sqrt{-g} \sum_{k=0}^n a_k(x) \times \frac{2^{2(n+1-k)} \pi^{1/2} (n+1-k)! \mathcal{B}_k}{2(4\pi)^{(2n+1)/2} (2n+2-2k)!}, \quad (24)$$

where

$$\mathcal{B}_k = (-1)^{n+1-k} \left(\sum_i m_{B_i}^{2n+1-2k} - \sum_i m_{F_i}^{2n+1-2k} \right) \quad (25)$$

are the renormalization constants. At the one-loop level the cosmological, gravitational, and coupling constants can be renormalized by

$$\frac{\Lambda}{8\pi G} - \frac{(n+1)! \mathcal{B}_0}{\pi^n (2n+2)!} = \frac{\Lambda^{\text{ren}}}{8\pi G^{\text{ren}}}, \\ \frac{1}{16\pi G} + \frac{n! \mathcal{B}_1}{4\pi^n (2n)!} = \frac{1}{16\pi G^{\text{ren}}}, \\ \frac{\alpha_1}{4\pi} + \frac{(n-1)! \mathcal{B}_2}{2^2 \times 144 \pi^n (2n-2)!} = \frac{\alpha_1^{\text{ren}}}{4\pi}, \\ \frac{\alpha_2}{4\pi} - \frac{(n-1)! \mathcal{B}_2}{2^3 \times 360 \pi^n (2n-2)!} = \frac{\alpha_2^{\text{ren}}}{4\pi}, \\ \frac{\alpha_3}{4\pi} + \frac{(n-1)! \mathcal{B}_2}{2^3 \times 360 \pi^n (2n-2)!} = \frac{\alpha_3^{\text{ren}}}{4\pi}. \quad (26)$$

We have seen that the effective action has different divergent structures depending on the parity of dimension and the renormalized effective action of even dimensions involves different renormalization constants from that of odd dimensions.

III. NUMBER OF STATES OF A MASSIVE SCALAR FIELD

We now turn to the thermodynamic entropy of the massive scalar field in a D -dimensional Schwarzschild and a nonextremal RN black hole. But there seems to be no simple and systematic method to evaluate the thermodynamic entropy of a quantum field in an arbitrary dimensional black hole background. We use directly the thermodynamic definition of entropy,

$$F = - \int_0^\infty dE \frac{g(E)}{e^{\beta E} - 1}, \quad (27)$$

where $g(E)$ is the number of states for a given E . In order to find the free energy of a scalar field it is necessary to find first the number of states.

The D -dimensional RN charged black hole ($D \geq 4$) has the metric [10]

$$ds^2 = -\Delta(r) dt^2 + \Delta^{-1}(r) dr^2 + r^2 d\Omega_{D-2}^2, \quad (28)$$

where

$$\Delta(r) = \left[1 - \left(\frac{r_-}{r} \right)^{D-3} \right] \left[1 - \left(\frac{r_+}{r} \right)^{D-3} \right], \quad (29)$$

where r_+ and r_- are the outer and inner event horizons given by

$$r_{\pm} = \left[\frac{4\Gamma((D-1)/2)}{(D-2)\pi^{(D-3)/2}} (M \pm \sqrt{M^2 - Q^2}) \right]^{1/(D-3)}, \quad (30)$$

where Γ is the gamma function. The Schwarzschild black hole is recovered as a limiting case of $Q=0$.

The number of states [6] of a massive scalar field in the black hole background (28) is given by

$$g_D(E, m) = \frac{1}{\pi} \int_{r_+ + h}^L \frac{dr}{\Delta(r)} \int dl (2l + D - 3) L_D(l) \\ \times \sqrt{E^2 - \left(m^2 + \frac{l(l+D-3)}{r^2} \right)} \Delta(r), \quad (31)$$

where

$$L_D(l) = \frac{\Gamma(l+D-3)}{\Gamma(D-2)\Gamma(l+1)} \quad (32)$$

is a multiplication factor of the degeneracy of angular momentum states. We note that the number of states in this form can be extended to a nonintegral dimension. By changing the variable

$$y = l(l+D-3), \quad (33)$$

we rewrite the number of states as

$$g_D(E, m) = \frac{1}{\pi} \int_{r_+ + h}^L \frac{dr}{r \sqrt{\Delta(r)}} \int_0^{y_+} dy L_D(l(y)) \sqrt{y_+ - y}, \quad (34)$$

where the integration is restricted to

$$y_+ = [E^2 - m^2 \Delta(r)] \frac{r^2}{\Delta(r)}. \quad (35)$$

We compute the number of states separately in even and odd dimensions.

A. Even dimensions

The multiplication factor of the degeneracy of angular momentum states in four dimensions is simply given by

$$L_4 = 1 \quad (36)$$

and, in $D = 2n$, ($n \geq 3$), by

$$\begin{aligned} L_{2n}(y) &= \frac{1}{\Gamma(2n-2)} \prod_{k=2}^{n-1} [y + (2n-k-2)(k-1)] \\ &\equiv \frac{1}{\Gamma(2n-2)} \sum_{k=0}^{n-2} C_k^{2n} y^k, \end{aligned} \quad (37)$$

whose coefficients are

$$C_{n-2}^{2n} = 1,$$

$$C_{n-3}^{2n} = \frac{1}{3}(n-2)(n-1)(2n-3),$$

The y integration [11] yields the number of states:

$$\begin{aligned} g_{2n}(E, m) &= \frac{1}{\pi \Gamma(2n-2)} \sum_{k=0}^{n-2} C_k^{2n} B(k+1, \frac{3}{2}) \\ &\times \int_{r_++h}^L \frac{dr}{r \sqrt{\Delta(r)}} y_+^{k+3/2}, \end{aligned} \quad (38)$$

where B is the beta function.

With the change of the variable

$$x = 1 - \left(\frac{r_+}{r} \right)^{2n-3},$$

$$\epsilon = (2n-3) \frac{h}{r_+}, \quad (40)$$

we obtain the number of states:

$$g_{2n}(E, m) = \frac{1}{\pi \Gamma(2n-2)} \sum_{k=0}^{n-2} C_k^{2n} B(k+1, \frac{3}{2}) \frac{r_+^{2k+3}}{2n-3} \int_{\epsilon}^1 dx \frac{[E^2 - x(1-u+ux)m^2]^{k+3/2}}{(1-x)^{1+(2k+3)/(2n-3)} x^{k+2} (1-u+ux)^{k+2}}, \quad (41)$$

where $u = (r_-/r_+)^{2n-3}$.

The most interesting ultraviolet divergences of free energy and entropy come from the part of the number of states that are the divergent functions of the brick wall thickness near the event horizon. So focusing on the lower limit of the integration, we expand the denominator around $x=0$:

$$\frac{1}{(1-x)^{1+(2k+3)/(2n-3)} (1-u+ux)^{k+2}} = \sum_{q=0}^{\infty} H_q^{2n,k} x^q, \quad (42)$$

where $H_q^{2n,k}$ are the coefficients of the Taylor expansion and are substituted into Eq. (41). To extract the divergent parts of Eq. (41), we transform the quadratic,

$$E^2 - x(1-u+ux)m^2 = \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right) Z \left[1 - \frac{u}{(1-u)^2 m^2 + 2uE^2} \left(Z + \frac{E^4}{Z} \right) \right], \quad (43)$$

where

$$Z = E^2 - (1-u)m^2 x, \quad (44)$$

and expand the numerator:

$$[E^2 - x(1-u+ux)m^2]^{k+3/2} = \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^{k+3/2} Z^{k+3/2} \sum_{p=0}^{\infty} (-1)^p \binom{k+\frac{3}{2}}{p} \left(\frac{u}{(1-u)^2 m^2 + 2uE^2} \right)^p \sum_{l=0}^p \binom{p}{l} E^{4l} Z^{p-2l}. \quad (45)$$

Finally we get the number of states:

$$g_{2n}(E, m) = \frac{1}{\pi \Gamma(2n-2)} \sum_{k=0}^{n-2} C_k^{2n} B\left(k+1, \frac{3}{2}\right) \frac{r_+^{2k+3}}{2n-3} \sum_{q=0}^{\infty} H_q^{2n,k} \sum_{p=0}^{\infty} (-1)^p \binom{k+\frac{3}{2}}{p}$$

$$\times \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^{k+3/2} \left(\frac{u}{(1-u)^2 m^2 + 2uE^2} \right)^p \sum_{l=0}^p \binom{p}{l} E^{4l} \int_{\epsilon} dx x^{-k-2+q} Z^{k+p-2l+3/2}, \quad (46)$$

whose integrations can be done relatively easily.

From the integral formulas in Appendix B, we see that the x integrals for $k+2 < q$ yield terms of the order of $O(1/m^2)$ and, for $k+2 > q$ and $k+p-2l+\frac{3}{2} < 0$, also lead to terms of the order of $O(1/m^2)$ even though they involve ultraviolet-divergent factors. Therefore all these integrals can be neglected in the large mass limit. The only divergent and nonvanishing terms come from the integrals with $k+2 > q$ and $k+p-2l+\frac{3}{2} > 0$. These integrals lead to the divergent structure

$$E^{2k+2p-6l+3} m^{2l} \frac{1}{\epsilon^{k+1-q-l}}, \quad l=0, 1, \dots, k-q, \quad (47)$$

and

$$E^{2p+2q-4l+1} m^{2k+2-2q} \ln(\epsilon). \quad (48)$$

We find the most divergent term of the number of states,

$$g_{2n}^{\text{m.div}} = \frac{1}{(n-1)(2n-3)\pi\Gamma(2n-2)} B(n-1, \frac{3}{2}) r_+^{2n-1} \times \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^{n-1/2} \frac{E^{2n-1}}{\epsilon^{n-1}}. \quad (49)$$

The most divergent term is used to give rise to Bekenstein-Hawking entropy in the brick wall model [6]. The most divergent term differs from that in Ref. [6]. Moreover, it should be noted that the most divergent term is to be removed in the Pauli-Villars regularization method as will be shown in the next sections.

B. Odd dimensions

We repeat the computation of the number of states in odd dimensions, which differs slightly from that in even dimensions. First, in five dimensions we have a quite simple multiplication factor of the angular momentum degeneracy:

$$L_5 = \frac{1}{2} \sqrt{y+1}. \quad (50)$$

In a general odd dimension $D=2n+1$ ($n \geq 3$), the number of states takes the form

$$L_{2n+1}(y) = \frac{1}{\Gamma(2n-1)} \sqrt{y+(n-1)^2} \times \prod_{k=2}^{n-1} [y+(2n-k-1)(k-1)] \\ \equiv \frac{1}{\Gamma(2n-1)} \sum_{k=0}^{n-2} C_k^{2n+1} y^k \sqrt{y+(n-1)^2}, \quad (51)$$

whose coefficients are

$$C_{n-2}^{2n+1} = 1,$$

$$C_{n-3}^{2n+1} = \frac{1}{6}(n-2)(n-1)(4n-3),$$

⋮

$$C_0^{2n+1} = \prod_{k=2}^{n-2} (2n-k-1)(k-1). \quad (52)$$

Note that the multiplication factor of the degeneracy of angular momentum states of odd dimensions has an additional factor of the square root which differs from that of even dimensions.

To get the number of states we do the intermediate y integration (34). The y integration can be done recursively:

$$J_k = \int_0^{y_+} dy y^k \sqrt{[y+(n-1)^2](y_+-y)} \\ = \frac{2k+1}{2(k+2)} [y_+ - (n-1)^2] J_{k-1} + \frac{k-1}{k+2} (n-1)^2 y_+ J_{k-2}, \quad (53)$$

whose two lowest integrals are, explicitly,

$$J_1 = \frac{(n-1)^3}{3} y_+^{3/2} + \frac{n-1}{8} [y_+ - (n-1)^2]^2 y_+^{1/2} \\ - \frac{1}{8} [y_+ - (n-1)^2] [y_+ + (n-1)^2]^2 \\ \times \left[\frac{\pi}{2} + \arcsin \left(\frac{y_+ - (n-1)^2}{y_+ + (n-1)^2} \right) \right], \\ J_0 = \frac{n-1}{4} [y_+ - (n-1)^2] y_+^{1/2} + \frac{1}{8} [y_+ + (n-1)^2]^2 \\ \times \left[\frac{\pi}{2} + \arcsin \left(\frac{y_+ - (n-1)^2}{y_+ + (n-1)^2} \right) \right]. \quad (54)$$

Near the event horizon ($y_+ \gg 1$) they are further approximated by

$$J_1 = \frac{(n-1)^3}{3} y_+^{3/2} + \frac{n-1}{8} [y_+ - (n-1)^2]^2 y_+^{1/2} \\ - \frac{1}{8} [y_+ - (n-1)^2] [y_+ + (n-1)^2]^2 \\ \times \left(\pi - 2(n-1)^2 \frac{1}{y_+^{1/2}} \right), \\ J_0 = \frac{n-1}{4} [y_+ - (n-1)^2] y_+^{1/2} + \frac{1}{8} [y_+ + (n-1)^2]^2$$

$$\times \left(\pi - 2(n-1)^2 \frac{1}{y_+} \right). \quad (55)$$

$$x = 1 - \left(\frac{r_+}{r} \right)^{2n-2}, \quad (58)$$

We can rearrange the number of states,

$$g_{2n+1}(E, m) = \frac{1}{\pi \Gamma(2n-1)} \sum_{k=0}^{n-2} C_k^{2n+1} \times \int_{r_++h}^L \frac{dr}{r \sqrt{\Delta(r)}} J_k(y_+), \quad (56)$$

as

$$g_{2n+1}(E, m) = \frac{1}{\pi \Gamma(2n-1)} \sum_{k=-1}^{2n} F_k^{2n+1} \int_{r_++h}^L \frac{dr}{r \sqrt{\Delta(r)}} y_+^{k/2}, \quad (57)$$

where F_k^{2n+1} is the coefficient of $y_+^{k/2}$ which is determined by solving the recursive relation (53) as a power series of y_+ . The number of states in odd dimensions differs from that in even dimensions by integral powers of y_+ .

Similarly as in the even dimensional case we change the variable

and replace a brick wall h by $\epsilon = (2n-2)h/r_+$. The radial integration of the number of states becomes

$$g_{2n+1}(E, m) = \frac{1}{\pi \Gamma(2n-1)} \sum_{k=-1}^{2n} F_k^{2n+1} \frac{r_+^k}{2n-2} \int_{\epsilon} dx \times \frac{[E^2 - x(1-u+ux)m^2]^{k/2}}{(1-x)^{1+k/(2n-2)} x^{(k+1)/2} (1-u+ux)^{(k+1)/2}}, \quad (59)$$

where $u = (r_-/r_+)^{2n-2}$.

Expanding the denominator around $x=0$,

$$\frac{1}{(1-x)^{1+k/(2n-2)} (1-u+ux)^{(k+1)/2}} = \sum_{q=0}^{\infty} K_q^{2n+1,k} x^q, \quad (60)$$

where $K_q^{2n+1,k}$ are the coefficients of the Taylor expansion, and the numerator

$$[E^2 - x(1-u+ux)m^2]^{k/2} = \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^{k/2} Z^{k/2} \sum_{p=0}^{\infty} (-1)^p \binom{k}{2} \left(\frac{u}{(1-u)^2 m^2 + 2uE^2} \right)^p \sum_{l=0}^p \binom{p}{l} E^{4l} Z^{p-2l}, \quad (61)$$

we finally get the number of states:

$$g_{2n+1}(E, m) = \frac{1}{\pi \Gamma(2n-1)} \sum_{k=-1}^{2n} F_k^{2n+1} \frac{r_+^k}{2n-2} \sum_{q=0}^{\infty} K_q^{2n+1,k} \sum_{p=0}^{\infty} (-1)^p \binom{k}{2} \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^{k/2} \left(\frac{u}{(1-u)^2 m^2 + 2uE^2} \right)^p \sum_{l=0}^p \binom{p}{l} E^{4l} \int_{\epsilon} dx x^{-(k+1)/2+q} Z^{k/2+p-2l}. \quad (62)$$

We may find the divergent structure of the number of states by doing directly the polynomial integrations for even integers k and computing the same integrals for odd integers k as in the even dimensional case. It is not difficult to find the most divergent term:

$$g_{2n+1}^{\text{mdiv}} = \frac{1}{2^{2n-1} (n-\frac{1}{2}) (2n-2) (n-1)! n!} r_+^{2n} \left(\frac{(1-u)^2 m^2 + 2uE^2}{(1-u)^2 m^2} \right)^n \frac{E^{2n}}{\epsilon^{n-1/2}}. \quad (63)$$

IV. FREE ENERGY IN THE FIVE- AND SIX-DIMENSIONAL RN BLACK HOLES

We work out explicitly the free energy and thermodynamic entropy of the scalar field in the five- and six-dimensional RN black hole backgrounds. These examples are enough to show the difference between even and odd dimensionality.

A. Five-dimensional RN black hole

We find explicitly the number of states in the five-dimensional RN black hole. Substituting Eq. (50) into Eq. (34), the exact number of states is found:

$$g_5(E, m) = \frac{1}{16\pi} \int_{r_++h}^L \frac{dr}{r \sqrt{\Delta(r)}} \left\{ 2(y_+-1) \sqrt{y_+} + (y_++1)^2 \left[\frac{\pi}{2} + \arcsin \left(\frac{y_+-1}{y_++1} \right) \right] \right\}. \quad (64)$$

Near the event horizon y_+ becomes large and we may approximate the number of states as

$$g_5(E, m) = \frac{1}{16} \int_{r_+ + h}^L \frac{dr}{r \sqrt{\Delta(r)}} \left(y_+^2 + 2y_+ - \frac{6}{\pi} y_+^{1/2} \right). \quad (65)$$

With the change of variable used in Sec. III, we rewrite the number of states as

$$g_5(E, m) = \frac{r_+^4}{32} \int_{\epsilon} dx \frac{[E^2 - x(1-u+ux)m^2]^2}{x^{5/2}(1-u+ux)^{5/2}(1-x)^3} + \frac{r_+^2}{16} \int_{\epsilon} dx \frac{[E^2 - x(1-u+ux)m^2]}{x^{3/2}(1-u+ux)^{3/2}(1-x)^2} - \frac{3r_+}{16\pi} \int_{\epsilon} dx \frac{[E^2 - x(1-u+ux)m^2]^{1/2}}{x(1-u+ux)(1-x)^{3/2}}. \quad (66)$$

After expanding the denominators of the integrations around $x=0$, doing integrations, and taking the limit of large mass, we obtain the number of states:

$$g_5(E, m) = \frac{r_+^4}{32} \left[\frac{2}{3(1-u)^{5/2}} \frac{E^4}{\epsilon^{3/2}} + \left(\frac{6-11u}{(1-u)^{7/2}} E^2 - \frac{4}{(1-u)^{3/2}} m^2 \right) \frac{E^2}{\epsilon^{1/2}} + O(\epsilon^{1/2}) \right] + \frac{r_+^2}{16} \left[\frac{2}{(1-u)^{3/2}} \frac{E^4}{\epsilon^{1/2}} + O(\epsilon^{1/2}) \right] - \frac{3r_+}{16\pi} \left[\frac{1}{(1-u)} A_{1/2}^1 - \frac{u}{2(1-u)^3 m^2} A_{3/2}^1 + O(\epsilon) + O\left(\frac{1}{m}\right) \right], \quad (67)$$

where $A_{(2p-1)/2}^q$ are integrals defined in Appendix B. The ultraviolet-divergent part of g_5 is

$$g_5^{\text{div}}(E, m) = \frac{r_+^4}{32} \left[\frac{2}{3(1-u)^{5/2}} \frac{E^4}{\epsilon^{3/2}} + \left(\frac{6-11u}{(1-u)^{7/2}} E^2 - \frac{4}{(1-u)^{3/2}} m^2 \right) \frac{E^2}{\epsilon^{1/2}} \right] + \frac{r_+^2}{16} \left[\frac{2}{(1-u)^{3/2}} \frac{E^4}{\epsilon^{1/2}} \right] + \frac{3r_+}{16\pi} \left[\frac{1}{(1-u)} E \ln(\epsilon) \right]. \quad (68)$$

The remaining part to be renormalized in the large mass limit is

$$g_5^{\text{ren}}(E, m) = \frac{3r_+}{16\pi} \left[\frac{2}{(1-u)} E + \frac{1}{(1-u)} E \ln \left(\frac{(1-u)m^2}{4E^2} \right) \right]. \quad (69)$$

From the definition (27), the free energy consists of two parts:

$$F_5(m) = F_5^{\text{div}}(m) + F_5^{\text{ren}}(m), \quad (70)$$

where the ultraviolet-divergent part is

$$F_5^{\text{div}}(m) = -\frac{r_+^4}{32} \left[\frac{2\Gamma(5)\zeta(6)}{3(1-u)^{5/2}\beta^5} \frac{1}{\epsilon^{3/2}} + \left(\frac{(6-11u)\Gamma(5)\zeta(5)}{(1-u)^{7/2}\beta^4} - \frac{4\Gamma(3)\zeta(3)}{(1-u)^{3/2}\beta^3} m^2 \right) \frac{1}{\epsilon^{1/2}} \right] - \frac{r_+^2}{16} \left[\frac{2\Gamma(5)\zeta(5)}{(1-u)^{3/2}\beta^5} \frac{1}{\epsilon^{1/2}} \right] - \frac{3r_+}{16\pi} \left[\frac{\Gamma(2)\zeta(2)}{(1-u)\beta^2} \ln(\epsilon) \right], \quad (71)$$

where ζ is the Riemann zeta function, and the part to be renormalized in the large mass limit is

$$F_5^{\text{ren}}(m) = -\frac{5r_+}{16\pi} \left[\frac{2\Gamma(2)\zeta(2)}{(1-u)\beta^2} + \frac{\Gamma(2)\zeta(2)}{(1-u)\beta^2} \ln \left(\frac{(1-u)m^2}{4\beta^2} \right) - \frac{\psi_1}{(1-u)\beta^2} \right], \quad (72)$$

where

$$\psi_k = 2 \int_0^\infty dt \frac{t^k \ln(t)}{e^t - 1}. \quad (73)$$

B. Six-dimensional RN black hole

From Sec. III we find the number of states in the six-dimensional RN black hole. After integrating the angular momentum states, the number of states is

$$g_6(E, m) = \frac{2r_+^5}{135\pi} \int_{\epsilon} dx \frac{[E^2 - x(1-u+ux)m^2]^{5/2}}{x^3(1-u+ux)^3(1-x)^{8/3}} + \frac{2r_+^3}{27\pi} \int_{\epsilon} dx \frac{[E^2 - x(1-u+ux)m^2]^{3/2}}{x^2(1-u+ux)^2(1-x)^2}. \quad (74)$$

Doing the x integration explicitly and keeping only divergent and nonvanishing terms for large masses ($m \gg 1$) of regulator fields, we obtain

$$g_6(E, m) = \frac{2r_+^5}{135\pi} \left[\left(\frac{1}{(1-u)^3} A_{5/2}^3 + \frac{8-17u}{3(1-u)^4} A_{5/2}^2 + \frac{44-160u+170u^2}{9(1-u)^5} A_{5/2}^1 \right) - \frac{5u}{2(1-u)^2 m^2} \right. \\ \times \left(\frac{1}{(1-u)^3} (A_{7/2}^3 + E^4 A_{3/2}^3) + \frac{8-17u}{3(1-u)^4} (A_{7/2}^2 + E^4 A_{3/2}^2) \right) + \frac{15u^2}{4(1-u)^7 m^4} (A_{9/2}^3 + 2E^4 A_{5/2}^3 + E^8 A_{1/2}^3) + O\left(\frac{1}{m}\right) \Big] \\ + \frac{2r_+^3}{27\pi} \left[\left(\frac{1}{(1-u)^2} A_{3/2}^2 + \frac{2-4u}{(1-u)^3} A_{3/2}^1 \right) - \frac{3u}{2(1-u)^4 m^2} (A_{5/2}^2 + E^4 A_{3/2}^2) + O\left(\frac{1}{m}\right) \right], \quad (75)$$

where $A_{(2p-1)/2}^q$ are integrals defined in Appendix B. We suppressed the terms that vanish as m goes to infinity. The number of states consists of the ultraviolet divergence terms

$$g_6^{\text{div}}(E, m) = \frac{2r_+^5}{135\pi} \left[\frac{1}{2(1-u)^3} \frac{E^5}{\epsilon^2} + \frac{32-23u}{12(1-u)^4} \frac{E^5}{\epsilon} + \left(\frac{95u}{8(1-u)^3} m^2 - \frac{3457-1120u-2740u^2}{72(1-u)^5} E^2 \right) E^3 \ln(\epsilon) \right] \\ + \frac{2r_+^3}{27\pi} \left[\frac{1}{(1-u)^2} \frac{E^3}{\epsilon} - \frac{4+u-9u^2}{2(1-u)^3} E^3 \ln(\epsilon) \right] \quad (76)$$

and the remaining terms

$$g_6^{\text{ren}}(E, m) = -\frac{2r_+^5}{135\pi} \left[\frac{23}{4(1-u)} m^4 E - \frac{368-547u}{6(1-u)^3} m^2 E^3 + \frac{16\,192+54\,400u-109\,715u^2}{1080(1-u)^5} E^5 \right. \\ \left. - \left(\frac{15}{8(1-u)} m^4 E - \frac{160-55u}{24(1-u)^3} m^2 E^3 + \frac{352+1120u-635u^2}{72(1-u)^5} E^5 \right) \ln\left(\frac{(1-u)m^2}{4E^2}\right) \right] \\ + \frac{2r_+^3}{27\pi} \left[\frac{4}{(1-u)} m^2 E - \frac{16+7u}{3} E^3 + \left(\frac{3}{2(1-u)} m^2 E - \frac{4+u}{2(1-u)^3} E^3 \right) \ln\left(\frac{(1-u)m^2}{4E^2}\right) \right]. \quad (77)$$

Likewise, the free energy consists of two parts:

$$F_6(m) = F_6^{\text{div}} + F_6^{\text{ren}}, \quad (78)$$

where the divergent part is

$$F_6^{\text{div}}(m) = -\frac{2r_+^5}{135\pi} \left[\frac{\Gamma(6)\zeta(6)}{2(1-u)^3 \beta^6} \frac{1}{\epsilon^2} + \frac{(32-23u)\Gamma(6)\zeta(6)}{12(1-u)^4 \beta^6} \frac{1}{\epsilon} \right. \\ \left. + \left(\frac{95u\Gamma(4)\zeta(4)}{8(1-u)^3 \beta^4} m^2 - \frac{(3457-1120u-2740u^2)\Gamma(6)\zeta(6)}{72(1-u)^5 \beta^6} \right) \ln(\epsilon) \right] \\ - \frac{2r_+^3}{27\pi} \left[\frac{\Gamma(4)\zeta(4)}{(1-u)^2 \beta^4} \frac{1}{\epsilon} - \frac{(4+u-9u^2)\Gamma(4)\zeta(4)}{2(1-u)^3 \beta^4} \ln(\epsilon) \right] \quad (79)$$

and the part to be renormalized is

$$F_6^{\text{ren}}(m) = \frac{2r_+^5}{135\pi} \left[\frac{23\Gamma(2)\zeta(2)}{4(1-u)\beta^2} m^4 - \frac{(368-547u)\Gamma(4)\zeta(4)}{6(1-u)^3 \beta^4} m^2 + \frac{(16\,192+54\,400u-109\,715u^2)\Gamma(6)\zeta(6)}{1080(1-u)^5 \beta^6} \right. \\ \left. + \left(\frac{15\Gamma(2)\zeta(2)}{8(1-u)\beta^2} m^4 - \frac{(160-55u)\Gamma(4)\zeta(4)}{24(1-u)^3 \beta^4} m^2 + \frac{(352+1120u-635u^2)\Gamma(6)\zeta(6)}{72(1-u)^5 \beta^6} \right) \ln\left(\frac{(1-u)m^2}{4\beta^2}\right) \right] \\ + \frac{(16\,192+54\,400u-109\,715u^2)\Gamma(6)\zeta(6)}{1080(1-u)^5 \beta^6} - \left(\frac{15\psi_1}{8(1-u)\beta^2} m^4 - \frac{(160-55u)\psi_3}{24(1-u)^3 \beta^4} m^2 + \frac{(352+1120u-635u^2)\psi_5}{72(1-u)^5 \beta^6} \right) \\ - \frac{2r_+^3}{27\pi} \left[\frac{4\Gamma(2)\zeta(2)}{(1-u)\beta^2} m^2 - \frac{(16+7u)\Gamma(4)\zeta(4)}{3\beta^4} + \left(\frac{3\Gamma(2)\zeta(2)}{2(1-u)\beta^2} m^2 - \frac{(4+u)\Gamma(4)\zeta(4)}{2(1-u)^3 \beta^4} \right) \ln\left(\frac{(1-u)m^2}{4\beta^2}\right) \right. \\ \left. - \left(\frac{3\psi_1}{2(1-u)\beta^2} m^2 - \frac{(4+u)\psi_3}{2(1-u)^3 \beta^4} \right) \right]. \quad (80)$$

V. PAULI-VILLARS REGULARIZATION

We now regularize the free energy using the Pauli-Villars regularization method used in Sec. II. We assume that the bosonic or fermionic regulator fields obey the same Bose-Einstein distribution [5]:

$$F_D(m) = \mp \int_0^\infty dE \frac{g_D(E, m)}{e^{\beta E} - 1}, \quad (81)$$

where the upper sign is for the bosonic fields and the lower sign for the fermionic fields.

The main idea of the Pauli-Villars regularization is to subtract the ultraviolet divergences of the original scalar field by those of regulator fields. To be more concrete we apply the Pauli-Villars regularization method to the thermodynamic entropy of the scalar field in five- and six-dimensional RN black hole backgrounds. We introduce a number of bosonic and fermionic regulator fields with masses m_{B_i} and m_{F_i} , respectively, whose number and mass conditions will be imposed later such that all the ultraviolet divergences and other unnecessary infinite quantities disappear. As mentioned earlier, the free energy of the fermionic regulator fields has the opposite sign from that of bosonic ones.

The total off-shell free energy of the scalar field in the five-dimensional RN black hole is the sum of those of bosonic and fermionic fields:

$$\begin{aligned} F_5 &= \sum_i F_5(m_{B_i}) - \sum_i F_5(m_{F_i}) \\ &= \sum_i F_5^{\text{div}}(m_{B_i}) - \sum_i F_5^{\text{div}}(m_{F_i}) + \sum_i F_5^{\text{ren}}(m_{B_i}) \\ &\quad - \sum_i F_5^{\text{ren}}(m_{F_i}), \end{aligned} \quad (82)$$

where $F_5(m)$ is given by Eq. (70). It can be shown easily that the ultraviolet divergences, which consist of $1/\epsilon^{3/2}$, $1/\epsilon$, $1/\epsilon^{1/2}$, and $\ln(\epsilon)$, may be removed, provided that the number of bosonic and fermionic fields are equal,

$$N_B = N_F = 3, \quad (83)$$

and the masses are required to satisfy

$$\sum_i^3 m_{B_i}^2 = \sum_i^3 m_{F_i}^2. \quad (84)$$

Note that these are the same conditions imposed to regularize the one-loop effective action in Sec. II. Then the remaining matter field contribution of the off-shell free energy is simply given by

$$F_5 = F_5^{\text{ren}} = \frac{3r_+}{16\pi} \frac{\Gamma(2)\zeta(2)}{(1-u)\beta^2} \mathcal{B}_2, \quad (85)$$

where

$$\mathcal{B}_2 = - \sum_{i=1}^3 \ln(m_{B_i}^2) + \sum_{i=1}^3 \ln(m_{F_i}^2). \quad (86)$$

It should be noted that the matter contribution (85) involves only $\mathcal{B}_2$, which is the renormalization constant (8) of six dimensions related to the coupling constant. The matter field contribution of the off-shell thermodynamic entropy, $S = \beta^2 \partial F / \partial \beta$, is given by

$$S_5^{\text{ren}} = - \frac{3r_+}{8\pi} \frac{\Gamma(2)\zeta(2)}{(1-u)\beta} \mathcal{B}_2. \quad (87)$$

Substituting the Hawking temperature

$$\beta_H = \frac{4\pi}{\Delta'(r_+)} = \frac{2\pi r_+}{1-u}, \quad (88)$$

one finds the matter field contribution of the on-shell entropy:

$$S_5^{\text{ren}} = - \frac{1}{32} \mathcal{B}_2. \quad (89)$$

We observe two important facts that are different from the four-dimensional case. First of all the thermodynamic entropy does not have a term proportional to the area of black hole event horizon,

$$A_5 = \frac{2\pi^2 r_+^3}{\Gamma(2)}, \quad (90)$$

in five dimensions. It is remarkable that the matter field contribution to the entropy in five dimensions does not involve the renormalization constant related to the gravitational constant, in strong contrast with the four-dimensional case, in which it was already observed that the renormalized thermodynamic entropy of a scalar field in the four-dimensional RN and the Schwarzschild black holes contributes a quantum correction to classical Bekenstein-Hawking entropy [5,12]. This means that in four dimensions through the renormalization of the gravitational constant the renormalization constant $\mathcal{B}_1$ renormalizes also Bekenstein-Hawking entropy

$$\frac{A_4}{4G} + \frac{\mathcal{B}_1 A_4}{12(4\pi)} = \frac{A_4}{4G^{\text{ren}}}, \quad (91)$$

so that the area law of black hole entropy is still valid even when one includes the matter field contribution of thermodynamic entropy. Second, we note that the matter field contribution of the thermodynamic entropy has a negative sign. A physical argument for these two facts is lacking at present.

We now turn to the six-dimensional RN black hole. The free energy is again the sum of those of bosonic and fermionic fields:

$$F_6 = \sum_i F_6(m_{B_i}) - \sum_i F_6(m_{F_i}). \quad (92)$$

There are divergences proportional to $1/\epsilon^2$, $1/\epsilon$, and $\ln(\epsilon)$. These ultraviolet divergences are removed by the conditions

$$N_B = N_F = 4 \quad (93)$$

and

$$\sum_{i=1}^4 m_{B_i}^2 = \sum_{i=1}^4 m_{F_i}^2. \quad (94)$$

Beside the ultraviolet-divergent terms there are also terms in Eq. (80) which become large as the masses of regulator fields become large. These terms can be removed by imposing an additional mass condition

$$\sum_{i=1}^4 m_{B_i}^4 = \sum_{i=1}^4 m_{F_i}^4. \quad (95)$$

With these conditions satisfied, one obtains the matter field contribution of the off-shell entropy:

$$\begin{aligned} S_6^{\text{ren}} = & \frac{2r_+^5}{135\pi} \left[\frac{15\Gamma(2)\zeta(2)}{4(1-u)\beta} \mathcal{B}_1 + \frac{(160-55u)\Gamma(4)\zeta(4)}{6(1-u)^3\beta^3} \mathcal{B}_2 \right. \\ & \left. + \frac{(352+1120u-635u^2)\Gamma(6)\zeta(6)}{12(1-u)^5\beta^5} \mathcal{B}_3 \right] \\ & + \frac{2r_+^3}{27\pi} \left[\frac{3\Gamma(2)\zeta(2)}{(1-u)\beta} \mathcal{B}_2 + \frac{2(4+u)\Gamma(4)\zeta(4)}{(1-u)^3\beta^3} \mathcal{B}_3 \right]. \end{aligned} \quad (96)$$

Substituting the Hawking temperature

$$\beta_H = \frac{4\pi r_+}{3(1-u)}, \quad (97)$$

one obtains the on-shell entropy

$$\begin{aligned} S_6^{\text{ren}} = & \frac{\Gamma(2)\Gamma\left(\frac{5}{2}\right)\zeta(2)}{12\pi^{9/2}} \mathcal{B}_1 \frac{A_6}{4} \\ & + r_+^2 \left(\frac{(160-55u)\Gamma(4)\zeta(4)}{64\pi^3} + \frac{\Gamma(2)\zeta(2)}{6\pi^2} \right) \mathcal{B}_2 \\ & + \left(\frac{(352+1120u-635u^2)\Gamma(6)\zeta(6)}{10 \cdot 240\pi^6} \right. \\ & \left. + \frac{(4+u)\Gamma(4)\zeta(4)}{16\pi^3} \right) \mathcal{B}_3, \end{aligned} \quad (98)$$

where

$$A_6 = \frac{2\pi^{5/2}r_+^4}{\Gamma(5/2)} \quad (99)$$

is the area of the six dimensional black hole event horizon. In fact we obtain the renormalized off-shell, Eq. (96), and on-shell, Eq. (98), contribution of the thermodynamic entropy by the quantum matter field which involves the same renormalization constants (8) used in the renormalization of the gravitational action. The first term gives rise to a quantum correction to the area law of the black hole.

Recollecting the renormalization of the gravitational constant in Eq. (9) in $D=6$ ($n=3$),

$$\frac{1}{16\pi G} + \frac{\mathcal{B}_1}{24(4\pi)^3} = \frac{1}{16\pi G^{\text{ren}}}, \quad (100)$$

we see that the total entropy of the scalar field, i.e., the sum of the bare Bekenstein-Hawking entropy and the quantum correction of entropy,

$$\frac{A_6}{4G} + \frac{\mathcal{B}_1 A_6}{24(4\pi)^2} = \frac{A_6}{4G^{\text{ren}}}, \quad (101)$$

renormalizes the Bekenstein-Hawking entropy. This implies that Bekenstein-Hawking entropy holds true at one loop in $D=6$ through the renormalization of the gravitational constant. The middle two terms proportional to r_+^2 are related to the renormalization of the one-loop quantum correction of gravity. The last two terms renormalize the cosmological constant.

VI. DISCUSSION

It should be remarked that the Pauli-Villars regularization method with all the regulator fields treated as obeying the Bose-Einstein or Fermi-Dirac statistics,

$$F_D = \mp \int_0^\infty dE \frac{g_D(E)}{e^{\beta E} \mp 1}, \quad (102)$$

depending on their spin statistics [12], may not work for the RN black hole in higher than four dimensions, because the ultraviolet-divergent structure has a hierarchy in which a higher dimensional black hole has divergent terms peculiar to that dimension and those also belonging to lower dimensional black holes. For instance in the five-dimensional RN black hole the ultraviolet-divergent part in Eq. (68) consists of terms proportional to E^4 , E^4 , E^2 , and E . But the Pauli-Villars regularization method with the correct spin statistics gives rise to the different statistical factors from

$$\begin{aligned} F_D = & \mp \int_0^\infty dE \frac{E^k}{e^{\beta E} \mp 1} \\ = & s_k^\mp \frac{\Gamma(k+1)\zeta(k+1)}{\beta^{k+1}}, \end{aligned} \quad (103)$$

where

$$s_k^\mp = \begin{pmatrix} 1, \\ 1 - \frac{1}{2^k} \end{pmatrix} \quad (104)$$

Thus to remove each divergent term with a different power of E one needs a different number of bosonic and fermionic fields. This means that all the ultraviolet-divergent terms cannot be removed at the same time by the Pauli-Villars regularization method with the correct spin statistics.

VII. CONCLUSION

In an attempt to understand how one might interpret the matter field contribution to the black hole entropy, we have studied the ultraviolet-divergent structures of a massive scalar field minimally coupled to a D -dimensional Reissner-Nordström black hole background and computed its thermodynamic entropy using the Pauli-Villars regularization method. We have computed the entropy with all the regula-

tor fields treated as bosons at finite temperature. As explicit examples we have elaborated the five- and six-dimensional cases in detail.

Interpreting the matter field contributions as the one-loop contribution to classical Bekenstein-Hawking entropy, we compared the resultant renormalization constants with those arising in the one-loop renormalization of the gravitational action. We have found that the matter field contribution does not, in general, yield the correct renormalization constants. In particular, we have found that, in an odd dimensional spacetime, the matter field contribution does not have a term proportional to the surface area of the event horizon. This result is consistent with the fact pointed out by DLM [5] that the correct interpretation of the matter field contribution as renormalizing the gravitational coupling constant is possible only when the scalar field coupling is minimal, even in four-dimensional spacetime.

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APPENDIX A: INTEGRALS USED IN PAULI-VILLARS REGULARIZATION

We derive the integral formulas useful in evaluating the effective action of the scalar field using the Pauli-Villars regularization method.

First, in an even dimension, we make use of the integral

$$\int_0^\infty dt e^{-zt} = \frac{1}{z}. \quad (\text{A1})$$

The integral (A1) is well defined for a complex z whose real part is positive, but can be continued analytically even to a pure imaginary z . We integrate the both sides of Eq. (A1) with respect to z from ζ_0 to z to obtain

$$\int_0^\infty dt \left(\frac{e^{-zt}}{t} - \frac{e^{-\zeta_0 t}}{t} \right) = -\ln(z) + \ln(\zeta_0), \quad (\text{A2})$$

and put $z=1$:

$$\int_0^\infty dt \left(\frac{e^{-t}}{t} - \frac{e^{-\zeta_0 t}}{t} \right) = \ln(\zeta_0). \quad (\text{A3})$$

By subtracting Eq. (A3) from Eq. (A2) and rearranging, we obtain

$$\int_0^\infty dt \frac{e^{-zt}}{t} = -\ln(z) + I_1, \quad (\text{A4})$$

where

$$I_1 = \int_0^\infty dt \frac{e^{-t}}{t}. \quad (\text{A5})$$

We repeat the integration of Eq. (A4) with respect to z from ζ_1 to z , and we get

$$\int_0^\infty dt \left(\frac{e^{-zt}}{t^2} - \frac{e^{-\zeta_1 t}}{t^2} \right) = z \ln(z) - z - \zeta_1 \ln(\zeta_1) + \zeta_1 - I_1(z - \zeta_1), \quad (\text{A6})$$

and put $z=0$ to the both sides of Eq. (A6) to get

$$\int_0^\infty dt \left(\frac{1}{t^2} - \frac{e^{-\zeta_1 t}}{t^2} \right) = -\zeta_1 \ln(\zeta_1) + \zeta_1 + I_1 \zeta_1. \quad (\text{A7})$$

Subtract Eq. (A7) from Eq. (A6) to obtain

$$\int_0^\infty dt \frac{e^{-zt}}{t^2} = z \ln(z) - z - z I_1 + I_2, \quad (\text{A8})$$

where

$$I_2 = \int_0^\infty dt \frac{1}{t^2}. \quad (\text{A9})$$

The n -times repetition of the integration leads to the integral formula

$$\begin{aligned} \int_0^\infty dt \frac{e^{-zt}}{t^n} &= \frac{(-1)^n}{(n-1)!} z^{n-1} \ln(z) + \frac{(-1)^n}{(n-1)!} \left(I_1 + \sum_{k=1}^{n-1} \frac{1}{k} \right) z^{n-1} \\ &\quad + \sum_{l=2}^n \frac{(-1)^{n-l}}{(n-l)!} I_l z^{n-l}, \end{aligned} \quad (\text{A10})$$

where

$$I_p = \int_0^\infty dt \frac{1}{t^p}, \quad (\text{A11})$$

for $p=2, 3, \dots$.

We can remove the infinite constants I_p in Eq. (A10) by adding and subtracting the integrals with different z . For instance,

$$\sum_{k=0}^N \int_0^\infty dt \left(\frac{e^{-z_{2k}t}}{t} - \frac{e^{-z_{2k+1}t}}{t} \right) = - \sum_{k=0}^N [\ln(z_{2k}) - \ln(z_{2k+1})] \quad (\text{A12})$$

and

$$\begin{aligned}
& \sum_{k=0}^N \int_0^\infty dt \left(\frac{e^{-z_{2k}t}}{t^2} - \frac{e^{-z_{2k+1}t}}{t^2} \right) \\
&= \sum_{k=0}^N [z_{2k} \ln(z_{2k}) - z_{2k+1} \ln(z_{2k+1})] \\
&\quad - I_1 \sum_{k=0}^N (z_{2k} - z_{2k+1}). \tag{A13}
\end{aligned}$$

We impose a condition

$$\sum_{k=0}^N z_{2k} = \sum_{k=0}^N z_{2k+1} \tag{A14}$$

to remove I_1 . In this way we obtain

$$\begin{aligned}
& \sum_{k=0}^N \int_0^\infty dt \left(\frac{e^{-z_{2k}t}}{t^n} - \frac{e^{-z_{2k+1}t}}{t^n} \right) \\
&= \frac{(-1)^n}{(n-1)!} \sum_{k=0}^N [z_{2k}^{n-1} \ln(z_{2k}) - z_{2k+1}^{n-1} \ln(z_{2k+1})], \tag{A15}
\end{aligned}$$

provided that

$$\sum_{k=0}^N z_{2k}^m = \sum_{k=0}^N z_{2k+1}^m \tag{A16}$$

for $m = 1, 2, \dots, n-1$. This is the main idea to get rid of the infinite quantities in the Pauli-Villars regularization method. We now may continue the integrals analytically even to a complex t , bearing in mind the subtraction procedure.

We can also obtain Eq. (A15) without introducing infinite constants I_p . We add and subtract Eq. (A2) with different z to get directly Eq. (A12). We integrate again Eq. (A2) with respect to z from ζ_1 to z to get

$$\begin{aligned}
& \int_0^\infty dt \left(\frac{e^{-zt}}{t^2} - \frac{e^{-\zeta_1 t}}{t^2} \right) + (z - \zeta_1) \int_0^\infty dt \frac{e^{-\zeta_1 t}}{t} \\
&= z \ln(z) - z[1 + \ln(\zeta_0)] - \zeta_1[\ln(\zeta_1) - \ln(\zeta_0) - 1]. \tag{A17}
\end{aligned}$$

By adding Eq. (A17) and subtracting with different z we obtain

$$\begin{aligned}
& \sum_{k=0}^N \int_0^\infty dt \left(\frac{e^{-z_{2k}t}}{t^2} - \frac{e^{-z_{2k+1}t}}{t^2} \right) \\
&= \sum_{k=0}^N [z_{2k} \ln(z_{2k}) - z_{2k+1} \ln(z_{2k+1})] \\
&\quad - \sum_{k=0}^N (z_{2k} - z_{2k+1})[1 + \ln(\zeta_0)]. \tag{A18}
\end{aligned}$$

We impose condition (A14) to get

$$\begin{aligned}
& \sum_{k=0}^N \int_0^\infty dt \left(\frac{e^{-z_{2k}t}}{t^2} - \frac{e^{-z_{2k+1}t}}{t^2} \right) \\
&= \sum_{k=0}^N [z_{2k} \ln(z_{2k}) - z_{2k+1} \ln(z_{2k+1})]. \tag{A19}
\end{aligned}$$

We can repeat the procedure n times to get Eq. (A15) provided that we impose conditions (A16). In either way we get identical regularized integrals.

Similarly, in an odd dimension, integrating both sides of the integral with respect to z ,

$$\int_0^\infty dt \frac{e^{-zt}}{t^{1/2}} = \frac{\pi^{1/2}}{z^{1/2}}, \tag{A20}$$

we obtain

$$\int_0^\infty dt \frac{e^{-zt}}{t^{3/2}} = -2\pi^{1/2} z^{1/2} + I_{3/2}. \tag{A21}$$

We repeat the integration with respect to z to get

$$\int_0^\infty dt \frac{e^{-zt}}{t^{5/2}} = \frac{4\pi^{1/2}}{3} z^{3/2} - I_{3/2} z + I_{5/2}. \tag{A22}$$

Repeating n times we get

$$\begin{aligned}
& \int_0^\infty dt \frac{e^{-zt}}{t^{(2n+1)/2}} = \frac{(-1)^n 2^{2n} \pi^{1/2} n!}{(2n)!} z^{(2n-1)/2} \\
&\quad + \sum_{l=1}^n \frac{(-1)^{n-l}}{(n-l)!} I_{(2l+1)/2} z^{n-l}. \tag{A23}
\end{aligned}$$

APPENDIX B: INTEGRAL FORMULA

In calculating the free energy we used frequently the following integral formula [11] for integrals q and p :

$$\int dx x^q Z^{(2p-1)/2}, \tag{B1}$$

where $Z(x) = E^2 - (1-u)m^2 x$. For positive integers q and p it becomes

$$\begin{aligned}
A_{(2p-1)/2}^q &= \int_{\epsilon} dx \frac{Z^{(2p-1)/2}(x)}{x^q} \\
&= Z^{(2p+1)/2}(\epsilon) \left[\frac{1}{(q-1)E^2\epsilon^{q-1}} + \sum_{l=1}^{q-2} (-1)^l \frac{(2p-2q+3)(2p-2q+5) \cdots (2p-2q+2l+1)}{2^l(q-1)(q-2) \cdots (q-l-1)\epsilon^{q-l-1}} \frac{1}{E^2} \left(\frac{(1-u)m^2}{E^2} \right)^l \right] \\
&\quad + (-1)^q \frac{(2p-2q+3)(2p-2q+5) \cdots (2p-3)(2p-1)}{2^{q-1}(q-1)!} \left(\frac{(1-u)m^2}{E^2} \right)^{q-1} \int_{\epsilon} dx \frac{Z^{(2p-1)/2}(x)}{x}, \\
A_{(2p-1)/2}^1 &= \int_{\epsilon} dx \frac{Z^{(2p-1)/2}(x)}{x} = -2 \sum_{l=1}^p \frac{E^{2(p-l)} Z^{(2l-1)/2}(\epsilon)}{2l-1} - E^{2p-1} \ln \left(\frac{E - Z^{1/2}(\epsilon)}{E + Z^{1/2}(\epsilon)} \right), \\
A_{1/2}^1 &= \int_{\epsilon} dx \frac{1}{x Z^{1/2}(x)} = -\frac{1}{E} \ln \left(\frac{E - Z^{1/2}(\epsilon)}{E + Z^{1/2}(\epsilon)} \right) \tag{B2}
\end{aligned}$$

and, also for positive integers q and p ,

$$\int_{\epsilon} dx x^q Z^{(2p-1)/2}(x) = (-1)^{q2} \left(\sum_{l=0}^q \frac{(-1)^l \binom{q}{l} E^{2l} Z^{q-l}(\epsilon)}{2q-2l+2p+1} \right) \frac{Z^{(2p+1)/2}(\epsilon)}{[(1-u)m^2]^{q+1}}. \tag{B3}$$

For integrals q and p we also have

$$\begin{aligned}
\int_{\epsilon} dx \frac{x^q}{Z^{p/2}(x)} &= \frac{\epsilon^q}{(1-p/2)(1-u)m^2 Z^{p/2-1}(\epsilon)} + \frac{q}{(1-p/2)(1-u)m^2} \int_{\epsilon} dx \frac{x^{q-1}}{Z^{p/2-1}(x)}, \\
\int_{\epsilon} dx \frac{1}{Z^{p/2}(x)} &= \frac{1}{(1-p/2)(1-u)m^2 Z^{p/2-1}(\epsilon)}. \tag{B4}
\end{aligned}$$

For $\epsilon \ll 1$, one has $Z(\epsilon) = E^2 + O(\epsilon)$. Keeping only divergent and nonvanishing terms as ϵ goes to zero, we get the following integrals frequently used in five and six dimensions:

$$\begin{aligned}
A_{1/2}^1 &= -2E - E \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{1/2}^2 &= \frac{E}{\epsilon} + \frac{(1-u)m^2}{E} + \frac{(1-u)m^2}{2E} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{1/2}^3 &= \frac{E}{2\epsilon^2} + \frac{(1-u)m^2}{4E\epsilon} + \frac{(1-u)^2 m^4}{4E^3} + \frac{(1-u)^2 m^4}{8E^3} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \tag{B5}
\end{aligned}$$

$$\begin{aligned}
A_{3/2}^1 &= -\frac{8E^3}{3} - E^3 \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{3/2}^2 &= \frac{E^3}{\epsilon} + 4(1-u)m^2 E + \frac{3(1-u)m^2 E}{2} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{3/2}^3 &= \frac{E^3}{2\epsilon^2} - \frac{(1-u)m^2 E}{4\epsilon} - \frac{(1-u)^2 m^4}{E} - \frac{3(1-u)^2 m^4}{8E} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \tag{B6}
\end{aligned}$$

$$\begin{aligned}
A_{5/2}^1 &= -\frac{46E^5}{15} - E^5 \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{5/2}^2 &= \frac{E^5}{\epsilon} + \frac{23(1-u)m^2 E^3}{3} + \frac{5(1-u)m^2 E^3}{2} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \\
A_{5/2}^3 &= \frac{E^5}{2\epsilon^2} - \frac{3(1-u)m^2 E^3}{4\epsilon} - \frac{23(1-u)^2 m^4 E}{4} - \frac{15(1-u)^2 m^4 E}{8} \ln \left(\frac{(1-u)m^2}{4E^2} \epsilon \right), \tag{B7}
\end{aligned}$$

$$\begin{aligned}
A_{7/2}^1 &= -\frac{352E^7}{105} - E^7 \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right), \\
A_{7/2}^2 &= \frac{E^7}{\epsilon} + \frac{176(1-u)m^2E^5}{15} + \frac{7(1-u)m^2E^5}{2} \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right), \\
A_{7/2}^3 &= \frac{E^7}{2\epsilon^2} - \frac{5(1-u)m^2E^5}{4\epsilon} - \frac{44(1-u)^2m^4E^3}{3} - \frac{35(1-u)^2m^4E^3}{8} \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right),
\end{aligned} \tag{B8}$$

and

$$\begin{aligned}
A_{9/2}^1 &= -\frac{1126E^9}{315} - E^9 \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right), \\
A_{9/2}^2 &= \frac{E^9}{\epsilon} + \frac{563(1-u)m^2E^7}{35} + \frac{9(1-u)m^2E^7}{2} \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right), \\
A_{9/2}^3 &= \frac{E^9}{2\epsilon^2} - \frac{7(1-u)m^2E^7}{4\epsilon} - \frac{563(1-u)^2m^4E^5}{20} - \frac{63(1-u)^2m^4E^5}{8} \ln\left(\frac{(1-u)m^2}{4E^2} \epsilon\right).
\end{aligned} \tag{B9}$$

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